SUMMING TO n

Abstract

We study nonnegative integer solutions of $x_1 + x_2 + \cdots + x_k = n$, where n and k are both fixed, and $x_1, \ldots, x_k$ are to be chosen.

Before we prove the theorem below, let's see a few examples. If $k = 1$, then we are looking for solutions to the equation $x_1 = n$. There's one solution to this silly equation. If $k = 2$, then we are looking for ordered pairs of nonnegative integers (x_1, x_2) such that $x_1 + x_2 = n$. There are $n + 1$ possibilities:

$$n + 0 = n, \quad (n-1) + 1 = n, \quad (n-2) + 2 = n, \quad \ldots, \quad 1 + (n-1) = n, \quad 0 + n = n.$$

If $k = 3$, then this gets more difficult. Consider the case $k = 3$ and $n = 2$; we are looking for ordered triples (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 2$. Here are the six possibilities:

$$\begin{array}{lll} 2+0+0=2, & 1+1+0=2, & 1+0+1=2, \\ 0+2+0=2, & 0+1+1=2, & 0+0+2=2. \end{array}$$

Theorem 1. *Suppose $k, n \in \mathbb{N}$ are fixed. Then the number of nonnegative integer solutions $(x_1, \ldots, x_k)$ to the equation*

$$x_1 + x_2 + \cdots + x_k = n$$

is equal to $\binom{n+k-1}{k-1}$.

Hint. Try to construct a bijection to sequences of red and green balls, as in an earlier proof, with red balls representing addition. You will need to use a bijection different from the one used in the earlier proof.

Corollary 2. *Suppose $k, n \in \mathbb{N}$. Then the number of **natural number** solutions $(x_1, \ldots, x_k)$ to the equation*

$$x_1 + x_2 + \cdots + x_k = n$$

is equal to $\binom{n-1}{k-1}$, if $n \geqslant k$, and is zero otherwise.

Hint. This is similar to the previous problem, except that each of the x_j have their minimal values increased by 1. Can you find some bijection to adjust the numbers in the previous theorem?