For $n \in \mathbb{N}$,

Let $AMGM(n)$ be the statement

"For every n-tuple of positive real numbers, $(x_1, x_2, \ldots, x_n)$,

$$\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{1}{n}\sum_{i=1}^{n} x_i ,$$

with equality iff $x_1 = x_2 = \ldots = x_n$."

__Theorem 1__ $\forall n \in \mathbb{N}$, $AMGM(n)$.

__Lemma 2__ $AMGM(2)$.

__Proof__ Let $\Delta = \sqrt{x_1} - \sqrt{x_2}$. Then $\Delta^2 \geq 0$ with equality iff $x_1 = x_2$.

$$\Rightarrow \quad 0 \leq \Delta^2 = \left(\sqrt{x_1} - \sqrt{x_2}\right)^2 = x_1 - 2\sqrt{x_1 x_2} + x_2$$

with equality iff $x_1 = x_2$

$$\Rightarrow \quad \sqrt{x_1 x_2} \leq \frac{x_1 + x_2}{2}$$ with equality iff $x_1 = x_2$. $\square$

__Lemma 3__ If $n \in \mathbb{N}$ and $AMGM(n)$ and $AMGM(2)$, then $AMGM(2n)$.

__Proof__

$$\sqrt[2n]{\prod_{i=1}^{n} x_i \prod_{i=1}^{n} y_i} = \sqrt{\sqrt[n]{\prod_{i=1}^{n} x_i}\;\sqrt[n]{\prod_{i=1}^{n} y_i}}$$

$$\leq \frac{1}{2}\left(\sqrt[n]{\prod_{i=1}^{n} x_i} + \sqrt[n]{\prod_{i=1}^{n} y_i}\right)$$

with equality iff $\sqrt[n]{\prod_{i=1}^{n} x_i} = \sqrt[n]{\prod_{i=1}^{n} y_i}$, by $AMGM(2)$.

$$\leq \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n} y_i\right)$$

with equality iff $\sqrt[n]{\prod_{i=1}^{n} x_i} = \sqrt[n]{\prod_{i=1}^{n} y_i}$ and $x_1 = x_2 = \dots = x_n$ and $y_1 = y_2 = \dots = y_n$, by AMGM (n) twice

$$= \frac{1}{2n}\left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\right) \qquad \text{(ie AM)}$$

and $\sqrt[n]{\prod_{i=1}^{n} x_i} = \sqrt[n]{\prod_{i=1}^{n} y_i}$ and $x_1 = x_2 = \dots = x_n$ and $y_1 = y_2 = \dots = y_n$

$\iff x_1 = x_2 = \dots = x_n = y_1 = y_2 = \dots = y_n$. $\square$

<u>Lemma 4</u> If $n \in \mathbb{N}$ and $n \geq 3$ and <u>AMGM(n)</u> then <u>AMGM$(n-1)$</u>.

<u>Proof</u> Suppose $(x_1, x_2, \dots, x_{n-1})$ is a sequence of $n-1$ positive reals, and let $x_n = \frac{x_1 + x_2 + \dots + x_{n-1}}{n-1}$. Then

$$\sqrt[n]{x_1 x_2 \dots x_{n-1} x_n} \leq \frac{x_1 + x_2 + \dots + x_{n-1} + x_n}{n}$$

with equality iff $x_1 = x_2 = \dots = x_{n-1} \cancel{= x_n}$ by AMGM(n)

$$= \frac{(x_1 + x_2 + \dots + x_{n-1})\left(1 + \frac{1}{n-1}\right)}{n} \qquad \frac{1+\frac{1}{n-1}}{n} = \frac{\cancel{n-1+1}}{\cancel{n}(n-1)}$$

$$= \frac{x_1 + x_2 + \dots + x_{n-1}}{n-1} = x_n$$

$$\Rightarrow \sqrt[n]{x_1 x_2 \ldots x_{n-1} x_n} \leq x_n$$ now divide both sides by $\sqrt[n]{x_n}$

$$\Rightarrow \sqrt[n]{x_1 x_2 \ldots x_{n-1}} \leq x_n^{\frac{n-1}{n}}$$ raise to power $\frac{n}{n-1}$

$$\Rightarrow \sqrt[n-1]{x_1 x_2 \ldots x_{n-1}} \leq x_n = \frac{x_1 + x_2 + \ldots + x_{n-1}}{n-1}.$$

with equality iff $x_1 = x_2 = \ldots = x_{n-1}$. $\square$