Lemma 1 If $k \in \mathbb{N}$ and S is a set with $\#S = n \geq k$, then

$$\# \mathcal{P}_k(S) = \frac{n-k+1}{k} \# \mathcal{P}_{k-1}(S)$$

Proof Define a kind of subset of S called a <u>subcommittee</u> which is nonempty $C \subset S$ s.t. one element of C is the <u>president</u>, denoted with a hat

Eg $S = \{a, b, c, d\}$, $\quad C = \{a, \hat{b}, c\}$

Denote set of all k-element subcommittees of S by $\hat{\mathcal{P}}_k(S)$.

Define $\kappa : \hat{\mathcal{P}}_k(S) \to \mathcal{P}_k(S)$ by making the president an ordinary element.

communism

$$\kappa(\{a, \hat{b}, c\}) = \{a, b, c\}$$

κ is k-to-1 ie k different elements of $\hat{\mathcal{P}}_k(S)$ give the same output.

$$\underbrace{\kappa(\{\hat{a}, b, c\}) = \kappa(\{a, \hat{b}, c\}) = \kappa(\{a, b, \hat{c}\})}_{k \text{ inputs}} = \underbrace{\{a, b, c\}}_{1 \text{ output.}}$$

Define $\rho : \hat{\mathcal{P}}_k(S) \to \mathcal{P}_{k-1}(S)$ by kicking out the president.

revolution

$$\rho(\{a, \hat{b}, c\}) = \underbrace{\{a, c\}}_{k-1}$$

ρ is $(n-k+1)$-to-1 because president could have been anything in S except for the elements that remain in the output.

$$\underbrace{\rho(\{a,c,\hat{d}\}) = \rho(\{a,\hat{b},c\})}_{n-(k-1)\text{ inputs}} = \underbrace{\{a,c\}}_{1\text{ output}}$$

$$k \underbrace{\#\mathcal{P}_k(S)}_{\text{outputs of }\kappa} = \underbrace{\#\widehat{\mathcal{P}}_k(S)}_{\text{inputs of }\kappa} = (n-(k-1)) \underbrace{\#\mathcal{P}_{k-1}(S)}_{\text{outputs of }\rho}$$

$$\Rightarrow \#\mathcal{P}_k(S) = \frac{n-k+1}{k} \#\mathcal{P}_{k-1}(S). \qquad \square$$