Lemma 1 If $k \in \mathbb{N}$ and S is a set with $\#S = n \geq k$, then

$$\# \mathcal{P}_k(S) = \frac{n-k+1}{k} \# \mathcal{P}_{k-1}(S)$$

Proof Define a kind of subset of S called a <u>subcommittee</u> which is nonempty $C \subset S$ s.t. one element of C is the <u>president</u>, denoted with a hat

Eg $S = \{a, b, c, d\}$, $C = \{a, \hat{b}, c\}$

Denote set of all k-element subcommittees of S by $\widehat{\mathcal{P}}_k(S)$.

Define $\kappa : \widehat{\mathcal{P}}_k(S) \to \mathcal{P}_k(S)$ by making the president an ordinary element.

communism

$$\kappa(\{a, \hat{b}, c\}) = \{a, b, c\}$$

κ is k-to-1 ie k different elements of $\widehat{\mathcal{P}}_k(S)$ give the same output.

$$\underbrace{\kappa(\{\hat{a}, b, c\}) = \kappa(\{a, \hat{b}, c\}) = \kappa(\{a, b, \hat{c}\})}_{k \text{ inputs}} = \underbrace{\{a, b, c\}}_{1 \text{ output.}}$$

Define $\rho : \widehat{\mathcal{P}}_k(S) \to \mathcal{P}_{k-1}(S)$ by kicking out the president.

revolution

$$\rho(\{a, \hat{b}, c\}) = \underbrace{\{a, c\}}_{k-1}$$

ρ is $(n-k+1)$-to-1 because president could have been anything in S except for the elements that remain in the output.

$$\underbrace{p(\{a,c,\hat{d}\}) = p(\{a,\hat{b},c\})}_{n-(k-1)\text{ inputs}} = \underbrace{\{a,c\}}_{1\text{ output}}$$

$$k\,\underbrace{\# \mathcal{P}_k(S)}_{\text{outputs of } \pi} = \underbrace{\# \hat{\mathcal{P}}_k(S)}_{\text{inputs of } \pi} = (n-(k-1))\,\underbrace{\# \mathcal{P}_{k-1}(S)}_{\text{outputs of } p}$$

$$\Rightarrow \# \mathcal{P}_k(S) = \frac{n-k+1}{k} \# \mathcal{P}_{k-1}(S).$$ $\square$

Theorem 2 If S is a finite set and $\# S = n$, then $\# \mathcal{P}_k(S) = \binom{n}{k}$.

Proof When $k=0$, $\# \mathcal{P}_0(S) = 1$.

$$\boxed{\begin{array}{l} S = \{a,b,c,d\} \\ \mathcal{P}_2(S) = \{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\} \end{array}}$$

Suppose $1 \le k \le n$ and suppose $\# \mathcal{P}_{k-1}(S) = \binom{n}{k-1}$. Then, by binomial thm,

$$\# \mathcal{P}_{k-1}(S) = \frac{n!}{(k-1)!\,(n-k+1)!}$$

$$\# \mathcal{P}_k(S) = \frac{n-k+1}{k} \# \mathcal{P}_{k-1}(S) \qquad \text{(by lemma 1)}$$

$$= \frac{\cancel{n-k+1}}{k} \cdot \frac{n!}{(k-1)!\,(n-k)!\,\cancel{(n-k+1)}}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k} \quad \text{by binomial thm.}$$

Hence, by induction on k, thm 2. $\square$

Corollary 3 If $\# S = n \in \mathbb{N}^0$, then $\# \mathcal{P}(S) = 2^n = 2^{\# S}$.

Proof $\mathcal{P}(S) = \mathcal{P}_0(S) \cup \mathcal{P}_1(S) \cup \mathcal{P}_2(S) \cup \dots \cup \mathcal{P}_n(S)$ is disjoint

$$\Rightarrow \# \mathcal{P}(S) = \# \mathcal{P}_0(S) + \# \mathcal{P}_1(S) + \mathcal{P}_2(S) + \dots + \mathcal{P}_n(S)$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n}$$

$$= \binom{n}{0} 1^0 1^n + \binom{n}{1} 1^1 1^{n-1} + \binom{n}{2} 1^2 1^{n-2} + \dots + \binom{n}{n} 1^n 1^0$$

$$= (1+1)^n = 2^n.$$

$\square$