Thm 1 If (V, E) is a tree then $\#V - \#E = 1$.

$$\boxed{\textbf{Thm 2} \text{ If } \Gamma = (V,E) \text{ is connected (with } \#V \geq 1) \text{ then any planar drawing of } \Gamma \text{ with } \#F \text{ faces obeys } \#V - \#E + \#F = 2.}$$

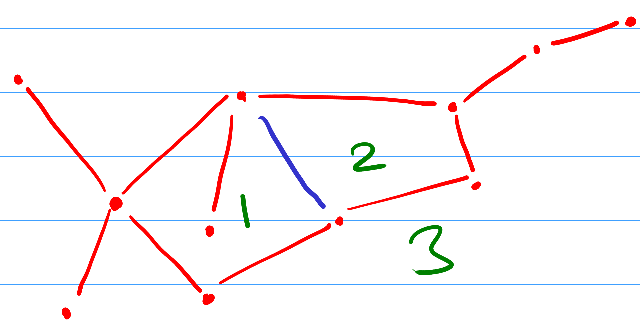
Proof We use induction on $\#E$ in Γ. If $\#E = 0$ then $\#V$ so the drawing is $\cdot$ so $\#V = 1$, $\#E = 0$, $\#F = 1$. ✓

Now assume the theorem has been proven for all graphs with $\#E = n-1$.

Consider a connected planar graph Γ with n edges, and a planar drawing of that graph.

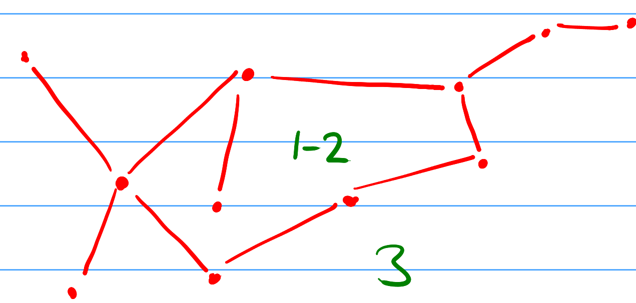
Case 1 Γ has no circuits. $\Rightarrow \Gamma$ is a tree $\Rightarrow$ (by thm 1) $\#V - \#E = 1$
$\Rightarrow \#F = 1$. ✓

Case 2 Γ has at least 1 circuit. Remove an edge from a circuit



$\Gamma = (V, E)$

F = set of faces



$\Gamma' = (V, E')$, $E' = E \setminus \{e\}$

F' = set of faces in

$\#F' = \#F - 1$

$\#V - \#E + \#F \stackrel{?}{=} 2$

$\#E' = n-1 \Rightarrow$ (ind. hyp)

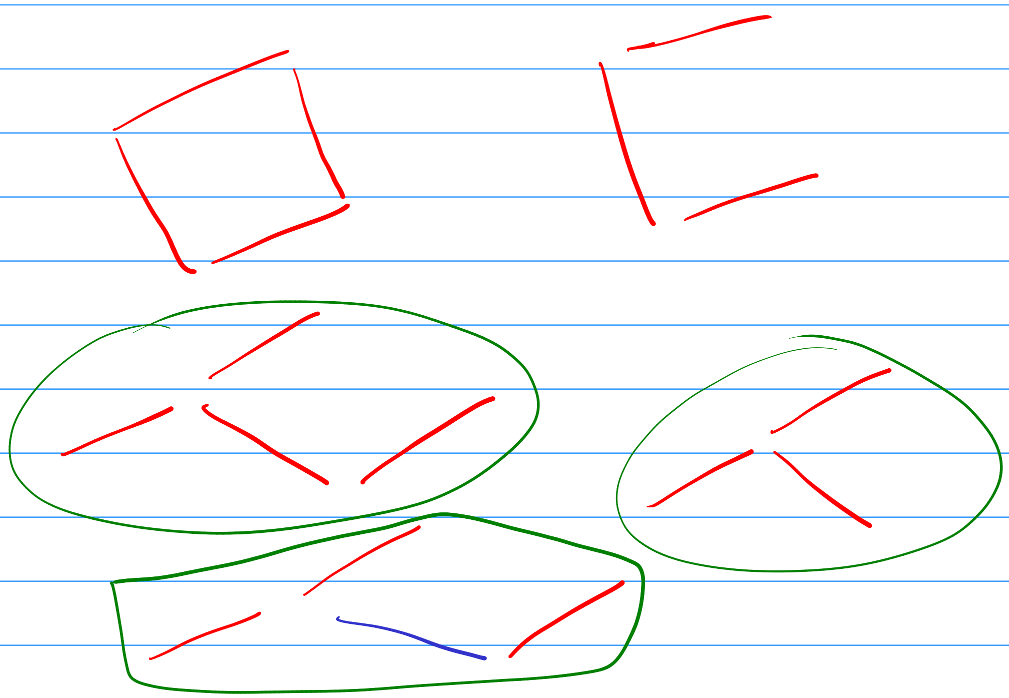
$$\#V - \underbrace{\#E'}_{\#E-1} + \underbrace{\#F'}_{\#F-1} = 2$$

$$\Rightarrow \#V - (\#E \cancel{-1}) + (\#F \cancel{-1}) = 2$$

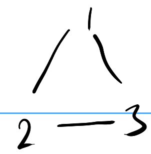
$$\Rightarrow \#V - \#E + \#F = 2.$$

So the theorem holds for our graph with $\#E = n$.

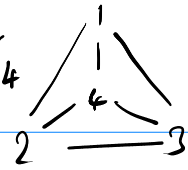
Hence, by induction on $\#E$, thm 2. $\square$



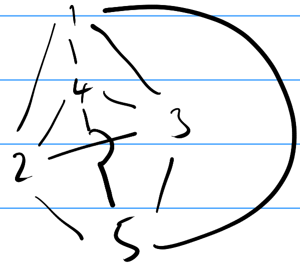
K_3



K_4



K_5



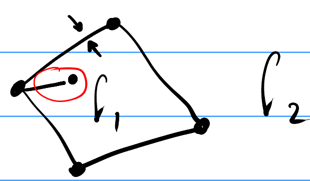
Lemma 1 If $\Gamma = (V, E)$ is connected planar graph with $\# V \geq 3$ then

$$\# E \leq 3 \# V - 6.$$

Proof $\# V \geq 3$. Suppose $\# E \leq 3$. Then $3\#V - 6 \geq 3(3) - 6 = 3 \geq \# E$.

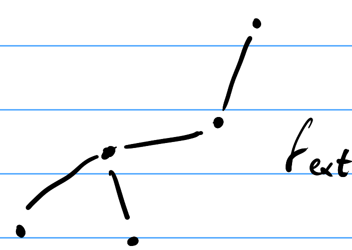
Henceforth assume $\# E \geq 4$.

For any face $f \in F$ define $\text{gon}(f) = \#$ of edges touching f.



$\text{gon}(f_1) = 5$
$\text{gon}(f_2) = 4$

every edge touches ≤ 2 faces $\Rightarrow \sum_{f \in F} \text{gon}(f) \leq 2 \# E$



every face touches ≥ 3 edges $\Rightarrow \sum_{f \in F} \text{gon}(f) \geq 3 \# F.$

$$3 \# F \leq 2 \# E$$

Thm 2 $\# V - \# E + \# F = 2$

$\Rightarrow 3\#V - 3\#E + 3\#F = 6$

$$\Rightarrow \quad 3\#V \underbrace{- 3\# + 2\#E} \geq 6$$

$$\Rightarrow \quad -\#E \geq 6 - 3\#V$$

$$\Rightarrow \quad \#E \leq 3\#V - 6 .$$

$\square$

Theorem 4 K_5 is not planar.

Proof $K_5 = (V, E)$ with $\#V = 5$, $\#E = \binom{5}{2} = \frac{5!}{2!\,3!} = 10$

But $\underline{10 \leq 3(5) - 6 = 9}$ is false. So, by lemma 3, K_5 is not planar.