Theorem 1 e is irrational.

Lemma 2 (Geometric Series) If $z \in \mathbb{R}$ and $|z| < 1$, then

$$1 + z + z^2 + \ldots = \frac{1}{1-z}.$$

Proof Let S_n denote the sum of the first $n+1$ terms (called a partial sum). Then $LHS = \lim_{n\to\infty} S_n$, and

$$S_n = 1 + z + z^2 + \ldots + z^n$$

so
$$\begin{aligned}(1-z)S_n &= 1 + z + z^2 + \ldots + z^n \\ &\quad - z - z^2 - \ldots - z^n - z^{n+1} \\ &= 1 - z^{n+1}\end{aligned}$$

Therefore, $S_n = \frac{1 - z^{n+1}}{1-z}$. As $|z| < 1$, $\lim_{n\to\infty} z^{n+1} = 0$.

But $$\lim_{n\to\infty} S_n = \frac{1 - \lim_{n\to\infty} z^{n+1}}{1 - z} = \frac{1}{1-z}. \qquad \square$$

Corollary 3 If $t \in \mathbb{R}$ and $|t| > 1$, then $\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \ldots = \frac{1}{t-1}$.

Proof Let $z = \frac{1}{t}$. Then $|z| < 1$. Hence, multiplying the eqn in Lemma 2 by z, we get $z + z^2 + z^3 + \ldots = \frac{z}{1-z}$

$$\Rightarrow \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \ldots = \frac{1/t}{1 - 1/t} = \frac{1}{t-1}. \qquad \square$$

Lemma 4 If $r > 0$, then

$$\frac{1}{r+1} < \frac{1}{r+1} + \frac{1}{(r+1)(r+2)} + \frac{1}{(r+1)(r+2)(r+3)} + \dots < \frac{1}{r}.$$

Proof $r > 0$, so each term in the middle series is positive. This justifies the first inequality.

$(r+1)(r+2)\ (r+1)^2$ $\qquad \frac{1}{(r+1)^2} > \frac{1}{(r+1)(r+2)}$

Note that $\rightarrow (r+1)(r+2) > (r+1)^2 > 0$,

$(r+1)(r+2)(r+3) > (r+1)^3 > 0$, etc.

Hence,
$$\frac{1}{r+1} + \frac{1}{(r+1)(r+2)} + \frac{1}{(r+1)(r+2)(r+3)} + \dots$$

$$< \frac{1}{r+1} + \frac{1}{(r+1)^2} + \frac{1}{(r+1)^3} + \dots$$

$$= \frac{1}{(r+1)-1} \quad \text{by corollary 3.}$$

$$= \frac{1}{r}.$$

Proof of Theorem 1 If $e \in \mathbb{Q}$, then $\exists\, n \in \mathbb{Z}$ s.t. $n > 1$ and $ne \in \mathbb{Z}$.

Define $x = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$,

$y = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots$ and note: $y > 0$.

Then $e = x + y$. So $n!e = n!x + n!y$.

But $n!e \in \mathbb{Z}$ and $n!x \in \mathbb{Z}$. So $n!y \in \mathbb{Z}$.

And $$n!\,y = \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + \dots$$

$$= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$

$$< \frac{1}{n}, \text{ by Lemma 4.}$$

So $n!\,y \in \mathbb{Z}$ and $0 < n!\,y < 1$, which is impossible.

So our original assumption was false; $e \notin \mathbb{Q}$. $\square$