$$+ : \mathbb{Z}^2 \to \mathbb{Z}$$

$$(a,b),\ (x,y) \in F = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$$

$$(a,b) \oplus (x,y) = (ay + bx,\ by) \qquad \mathbb{Q} = F/\sim$$

$$\begin{tikzcd}
(a_1,a_2),\ (b_1,b_2) \in F \arrow[rr, "\substack{\text{apply the binary operator} \\ \oplus : F \times F \to F}"] & & (a_1 b_2 + a_2 b_1,\ a_2 b_2) \in F \arrow[d, "\substack{\text{apply canonical} \\ \text{projection}}"] \\
\alpha, \beta \in \mathbb{Q} = F/\sim \arrow[u, "\substack{\text{pick representatives} \\ (a_1,a_2) \in \alpha,\ (b_1,b_2) \in \beta}"] \arrow[rr, "\text{defined by red arrows}", "\substack{\text{apply the binary operator} \\ \overline{\oplus} : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}}"'] & & \alpha \overline{\oplus} \beta \in \mathbb{Q}
\end{tikzcd}$$

Field axiom 2

Proof WTS $\alpha \overline{\oplus} \beta = \beta \overline{\oplus} \alpha$. $(a_1, a_2) \in \alpha$, $(b_1, b_2) \in \beta$

$$\alpha \overline{\oplus} \beta \ni (a_1 b_2 + a_2 b_1,\ a_2 b_2) = (b_1 a_2 + b_2 a_1,\ b_2 a_2) \in \beta \overline{\oplus} \alpha$$

$$\Rightarrow \alpha \overline{\oplus} \beta = \beta \overline{\oplus} \alpha. \quad \square$$

$$(a,b) \otimes (x,y) = (ax,\ by)$$

$$\downarrow$$

$$\alpha \overline{\otimes} \beta$$