Lemma 1 (Triangle inequality) Suppose $\alpha, \beta \in \mathbb{Q}$. Then $|\alpha+\beta| \leq |\alpha| + |\beta|$.

Proof It is equivalent to prove $(\alpha+\beta)^2 \leq (|\alpha| + |\beta|)^2$

$$\text{RHS} = |\alpha|^2 + 2|\alpha\beta| + |\beta|^2 = \alpha^2 + 2|\alpha\beta| + \beta^2$$

$$\text{LHS} = \alpha^2 + 2\alpha\beta + \beta^2$$

$$\text{LHS} \leq \text{RHS}.$$

$\square$

Proposition 2 Suppose $(a_n)_{n\in\mathbb{N}}$ is a convergent rational sequence.
Then $(a_n)_{n\in\mathbb{N}} \in C_{\mathbb{Q}}$.

Proof $(a_n)_{n\in\mathbb{N}}$ is rational by hypothesis. So we only need to prove it is Cauchy.
$(a_n)_{n\in\mathbb{N}}$ is convergent to $a \in \mathbb{Q}$

$$\Rightarrow \lim_{n\to\infty} (a_n - a) = 0$$

N′ may depend on δ, so we write $N'(\delta)$

$$\Rightarrow \forall \delta > 0 \; \exists N' \in \mathbb{N} \text{ s.t. } \forall n > N', \; |a_n - a| < \delta$$

$\delta = 1$: $N'(1) = 5$

$\delta = \frac{1}{100}$: $N'(\frac{1}{100}) = 48$ $|a_{49} - a| < \frac{1}{100}, \; |a_{50} - a| < \frac{1}{100}, \ldots$

[WTS $\forall \varepsilon > 0 \; \exists \underline{N} \in \mathbb{N}$ s.t. $\forall m, n > N, \; |a_m - a_n| < \varepsilon$.]

ie $N(\varepsilon) = N'(\frac{\varepsilon}{2})$

Fix any $\varepsilon > 0$. $\underline{\forall m, n > N'(\frac{\varepsilon}{2})}$, $|a_n - a| < \frac{\varepsilon}{2}$, $|a_m - a| < \frac{\varepsilon}{2}$

Consider
$$|a_m - a_n| = |(a_m - a) + (a - a_n)|$$
$$\leq |a_m - a| + |a_n - a|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow |a_m - a_n| < \varepsilon.$$

$\square$