Theorem 1 e is irrational.

Lemma 2 (Geometric Series Lemma) If $z \in \mathbb{R}$ and $|z|<1$, then

$$1 + z + z^2 + \dots = \frac{1}{1-z}.$$

Proof Let S_n denote sum of first $n+1$ terms of LHS (called a partial sum). Then LHS $= \lim_{n\to\infty} S_n$.

$$S_n = 1 + z + z^2 + \dots + z^n$$

so
$$\begin{aligned}(1-z)S_n &= \left(1 + \cancel{z} + \cancel{z^2} + \cancel{\dots} + \cancel{z^n}\right) \\ &\quad - \left(\cancel{z} + \cancel{z^2} + \cancel{\dots} + \cancel{z^n} + z^{n+1}\right) \\ &= 1 - z^{n+1}.\end{aligned}$$

Therefore $S_n = \dfrac{1-z^{n+1}}{1-z}$.

As $|z|<1$, $\lim_{n\to\infty} z^{n+1} = 0$. But

$$\text{LHS} = \lim_{n\to\infty} S_n = \frac{1-\lim_{n\to\infty} z^{n+1}}{1-z} = \frac{1}{1-z} = \text{RHS}. \qquad \square$$

Corollary 3 If $t \in \mathbb{R}$ and $|t|>1$, then $\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \dots = \frac{1}{t-1}$.

Proof Multiply previous lemma by z:

$$z + z^2 + z^3 + \dots = \frac{z}{1-z}, \quad \text{for } z \in \mathbb{R} \text{ with } |z|<1$$

Let $\boxed{t \in \mathbb{R} \text{ with } |t| > 1}$ let $z = \frac{1}{t}$

$$\Rightarrow \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \dots = \frac{\frac{1}{t}}{1 - \frac{1}{t}} = \frac{1}{t-1}. \qquad \square$$

Lemma 4 If $r > 0$, then

$$\frac{1}{r+1} < \frac{1}{r+1} + \frac{1}{(r+1)(r+2)} + \frac{1}{(r+1)(r+2)(r+3)} + \dots < \frac{1}{r}.$$

Proof $r > 0$, so each term in middle series is positive. This justifies first inequality. Note that

$$(r+1)(r+2) > (r+1)^2 > 0,$$
$$(r+1)(r+2)(r+3) > (r+1)^3 > 0, \quad \text{etc.}$$

Hence
$$\frac{1}{r+1} + \frac{1}{(r+1)(r+2)} + \frac{1}{(r+1)(r+2)(r+3)} + \dots$$
$$< \frac{1}{r+1} + \frac{1}{(r+1)^2} + \frac{1}{(r+1)^3} + \dots$$
$$= \frac{1}{(r+1)-1} = \frac{1}{r} \quad \text{by corollary 3.} \qquad \square$$

Proof of theorem 1 If $e \in \mathbb{Q}$, then $\exists\, n \in \mathbb{Z}$ s.t. $n > 1$ and $ne \in \mathbb{Z}$.

Define
$$x = \frac{1}{0!} + \frac{1}{1!} + \dots + \frac{1}{(n-1)!} + \frac{1}{n!}$$
$$y = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots \qquad \text{and note } y > 0.$$

Then $e = x + y$. So $n!e = n!x + n!y$. But $n!e \in \mathbb{Z}$ and $n!x \in \mathbb{Z}$

So $n!y \in \mathbb{Z}$.

But $$n!\,y = \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + \frac{n!}{(n+4)!} + \frac{n!}{(n+5)!} + \cdots$$

$$= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \frac{1}{(n+1)\cdots(n+5)} + \cdots$$

$$< \frac{1}{n} \leq \frac{1}{2} < 1.$$

So $n!\,y \in \mathbb{Z}$ and $0 < n!\,y < 1$, which is impossible.

Therefore, our original assumption was false; $e \notin \mathbb{Q}$. $\square$