$K_{3,3}$ is not planar.

Definition If F is the set of faces of a planar drawing of a graph, and $f \in F$, then $\text{gon}(f)$ is number of edges touch f.

Definition A bipartite graph has its vertices are divided into two sets U_1, U_2, and edges join $v \in U_1$ to $w \in U_2$, never $v \in U_1$ to $w \in U_1$ nor $v \in U_2$ to $w \in U_2$.

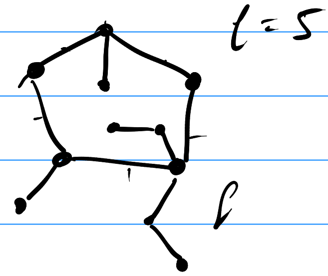
Lemma Suppose $\Gamma = (V, E)$ is planar graph. Moreover, suppose $\# E \geq \ell > 2$ and Γ has no circuit of length $< \ell$. Then

$$\# E \leq \frac{\ell}{\ell - 2} (\# V - 2).$$

Proof Consider planar drawing of Γ, with face set F.

$$\sum_{f \in F} \text{gon}(f) \leq 2 \# E \qquad \text{(see proof of } K_5 \text{ nonplanar)}$$

Every bounded face must touch at least ℓ edges
Exterior face must touch at least ℓ edges too



$$\Rightarrow \quad \sum_{f \in F} \text{gon}(f) \geq \ell \# F$$

$$\Rightarrow \quad \ell \# F \leq 2 \# E$$

Euler's formula $\#V - \#E + \#F = 2$

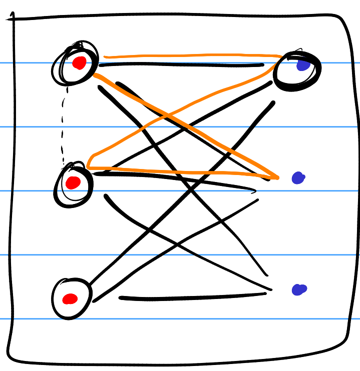
$$\Rightarrow \qquad \ell \#F = \underbrace{2\ell + \ell\#E - \ell\#V}_{} \leq 2\#E$$

$$\Rightarrow \qquad (\ell - 2)\#E \leq \ell(\#V - 2)$$

$$\Rightarrow \qquad \#E \leq \frac{\ell}{\ell - 2}(\#V - 2). \qquad \square$$

Theorem $K_{3,3}$ is not planar.

Proof



$\#E = 9, \quad \#V = 6$

$\ell = 4.$

So, by the lemma $9 \leq \frac{4}{4-2}(6-2) = 8$,

which is false. So $K_{3,3}$ is not planar. $\square$