Counting Permutations

Definition $a \in X \neq \emptyset$, bijection $f: X \to X$ is a **permutation**

$\text{Perm}(X)$ = set of all permutations on X

$\text{Perm}_a(X)$ = set of all permutations on X which send a to a.

Lemma 2 $\#X = n > 1$, $a \in X$, then

$$\#\text{Perm}_a(X) = \#\text{Perm}(X \setminus \{a\}).$$

Proof Define $s: \text{Perm}(X \setminus \{a\}) \longrightarrow \text{Perm}_a(X)$.

$$[s\phi](x) = \begin{cases} a & \text{if } x = a, \\ \phi(x) & \text{otherwise.} \end{cases}$$

Claim: s is bij.

Inj $\phi, \psi \in \text{Perm}(X \setminus \{a\})$ and $\phi \neq \psi$. Then $\exists\, x \in X \setminus \{a\}$ s.t. $\phi(x) \neq \psi(x)$. Then $[s\phi](x) = \phi(x) \neq \psi(x) = [s\psi](x)$.

$$\Rightarrow [s\phi](x) \neq [s\psi](x).$$

Hence s is injective.

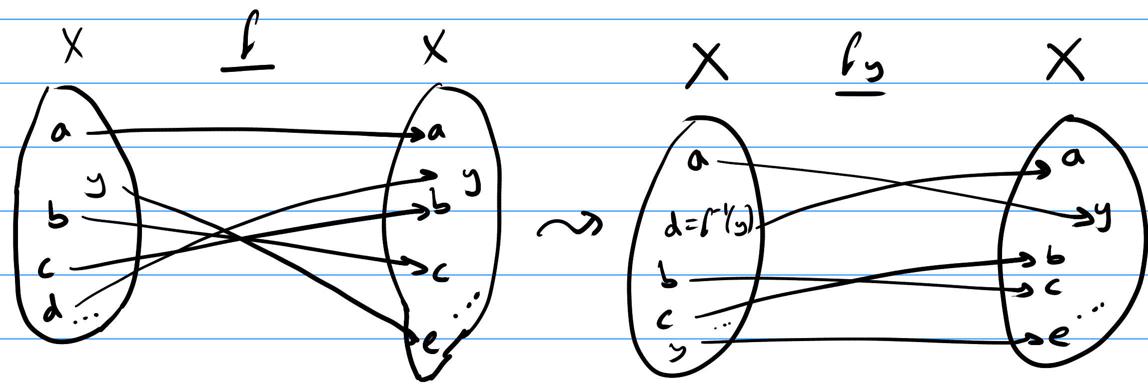
Surj. Consider $f \in \text{Perm}_a(X)$. We define ϕ by $\phi(x) = f(x)$ for all $x \in X \setminus \{a\}$. Then $\phi: X \setminus \{a\} \to X \setminus \{a\}$ and it is a bijection because f was. But $s\phi = f$. Hence s is surjective.

So s is a bijection. Hence $\#\text{Perm}_a(X) = \#\text{Perm}(X \setminus \{a\})$. $\square$

Lemma 3 Suppose X is finite set, $a, y, z \in X$. For $f, g \in \mathrm{Perm}_a(X)$, define $f_y, g_z \in \mathrm{Perm}(X)$ by

$$f_y(x) = \begin{cases} y & \text{if } x = a \\ a & \text{if } x = f^{-1}(y) \\ f(x) & \text{otherwise} \end{cases} \qquad g_z(x) = \begin{cases} z & \text{if } x = a \\ a & \text{if } x = g^{-1}(z) \\ g(x) & \text{otherwise} \end{cases}$$

Then $f_y = g_z$ if and only if $f = g$ and $y = z$.



Lemma 4 X finite set, $a \in X$, $h \in \mathrm{Perm}(X)$. Then $\exists\, y \in X$ and $f \in \mathrm{Perm}_a(X)$ s.t. f_y (as defined above) satisfies $f_y = h$.

Theorem 5 If X is finite with $\#X = n \in \mathbb{N}$ then $\#\mathrm{Perm}(X) = n!$

Proof attempt 1 $\#X = n \in \mathbb{N}$. Consider $f \in \mathrm{Perm}(X)$

Info:

- The first entry of X must be mapped to any ~~other~~ entry of X so there are n ways to do this.
- The second can be mapped to any of the other $n-1$.
- The 3rd ,, $n-2$.

$\vdots$ ← hides an induction

- The last $\quad \ldots \quad 1$

Overall we have choices $\quad n(n-1)(n-2)\ldots 1 = n!$

Proof attempt 2 Suppose $\#X = 1$. Then there is only permutation on X: the identity. $n=1 \Rightarrow n! = 1! = 1$.

Suppose $\exists\, n > 1$, for all sets X with $\#X = n-1$, $\#\mathrm{Perm}(X) = (n-1)!$

Consider X with $\#X = n$. WTS $\#\mathrm{Perm}(X) = n!$

$a \in X$. If $f \in \mathrm{Perm}(X)$, then there are n choices for y s.t. $f(a) = y$

$\mathrm{rest}_f : X \setminus \{a\} \longrightarrow X \setminus \{y\} \qquad \# X\setminus\{a\} = n-1 = \# X\setminus\{y\}$

By ind. hyp. $\mathrm{rest}_f \in \mathrm{Bij}(X\setminus\{a\}, X\setminus\{y\})$ can be any of $(n-1)!$ functions. So f can be any of $n \times (n-1)! = n!$

Hence, by induction on n, Theorem 5. $\quad \boxtimes$

Proof attempt 3 Suppose $\#X = 1$. Then there is only permutation on
Base: X: the identity. $n=1 \Rightarrow n! = 1! = 1$.

Ind. Hyp. Suppose $\exists\, n > 1$, for all sets X with $\#X = n-1$, $\#\mathrm{Perm}(X) = (n-1)!$

Consider X with $\#X = n$. WTS $\#\mathrm{Perm}(X) = n!$

$a \in X$. Consider $X \setminus \{a\}$. Then, by ind. hyp., $\#\mathrm{Perm}(X \setminus \{a\}) = (n-1)!$

How do we "add a back in to the domain & codomain of each $f \in \mathrm{Perm}(X \setminus \{a\})$" ?

From $f \in \mathrm{Perm}(X \setminus \{a\})$ produce a function in $\mathrm{Perm}_a(X)$

And (I think!) $\#\mathrm{Perm}(X \setminus \{a\}) = \#\mathrm{Perm}_a(X)$.

Idea! I think I can "swap" the a in the codomain of each function in $\mathrm{Perm}_a(X)$ with any element of the codomain. There are n ways to do so.

This process is $t : \mathrm{Perm}_a(X) \times X \longrightarrow \mathrm{Perm}(X)$.

$$t(f, y) = f_y$$

I think t is a bijection.

If it is then $\#(\mathrm{Perm}_a(X) \times X) = \#\mathrm{Perm}(X)$

and
$$\begin{aligned}\#(\mathrm{Perm}_a(X) \times X) &= \#\mathrm{Perm}_a(X) \times \#X \\ &= (n-1)! \times n \\ &= n!\end{aligned}$$

Hence, by induction on n, Theorem 5. $\square$

Proof attempt 2' Suppose $\#X = 1 = \#Y$. Then there is only ~~permutation on~~ bijection from X to Y

Base ~~X = the identity.~~ $n = 1 \Rightarrow n! = 1! = 1$.

Ind hyp Suppose $\exists\, n > 1$, for all sets X, Y with $\#X = n-1 = \#Y$, $\cancel{\#\mathrm{Perm}(X)}\ \#\mathrm{Bij}(X,Y) = (n-1)!$

Consider X, Y with $\#X = n = \#Y$. WTS $\cancel{\#\mathrm{Perm}(X)}\ \#\mathrm{Bij}(X,Y) = n!$

$a \in X$. If $f \in \cancel{\mathrm{Perm}(X)}\ \mathrm{Bij}(X,Y)$, then there are n choices of $y \in Y$ s.t. $f(a) = y$.

$\mathrm{rest}_{\text{of } f} : X \setminus \{a\} \longrightarrow \cancel{X \setminus \{y\}}\ Y \setminus \{y\}$ $\quad \#X \setminus \{a\} = n-1 = \cancel{\#X \setminus \{y\}}\ \#Y \setminus \{y\}$

By ind. hyp. $\mathrm{rest}_{\text{of } f} \in \mathrm{Bij}(X \setminus \{a\}, \cancel{X \setminus \{y\}}\ Y \setminus \{y\})$ can be any of $(n-1)!$ functions. So f can be any of $n \times (n-1)! = n!$

Hence, by induction on n, Theorem $\cancel{5}$ 6. $\square$

Theorem 6 If X, Y is finite with $\#X = n \in \mathbb{N}$, $= \#Y$ then $\cancel{\#\mathrm{Perm}(X)}\ \#\mathrm{Bij}(X,Y) = n!$

Proof Theorem 5 Theorem 6, with $Y = X$. $\square$