Compositions of n

Definition 1 $n \in \mathbb{N}$ composition of n is ordered sequence of $\mathbb{N}$ that sums to n. eg, $n=4$

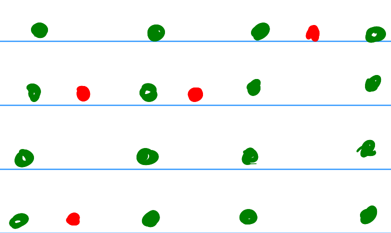
$$\begin{cases} \underset{x_1}{4} = 4 & \underset{x_1+x_2}{3+1} = 4 & \underset{x_1+x_2}{1+3} = 4 & 2+2=4 \\ 2+1+1=4 & 1+2+1=4 & \underset{x_1+x_2+x_3}{1+1+2} = 4 & 1+1+1+1=4 \end{cases}$$

Theorem 2 there are 2^{n-1} compositions of n

Proof $n \in \mathbb{N}$.

Bijection: compositions of n $\longrightarrow$ Ball patterns. Defined by

Arrange n green balls in a row with slots between each. Insert a red ball after x_1 green balls. Another red ball after another x_2 green balls, etc.



$3+1=4$	$\{3\}$
$1+1+2=4$	$\{1,2\}$
$4=4$	$\{\ \}$
$1+3=4$	$\{1\}$

f, Bijection: $\mathcal{P}(\{1,2,3,\dots,n-1\}) \longrightarrow$ ball patterns. Defined by

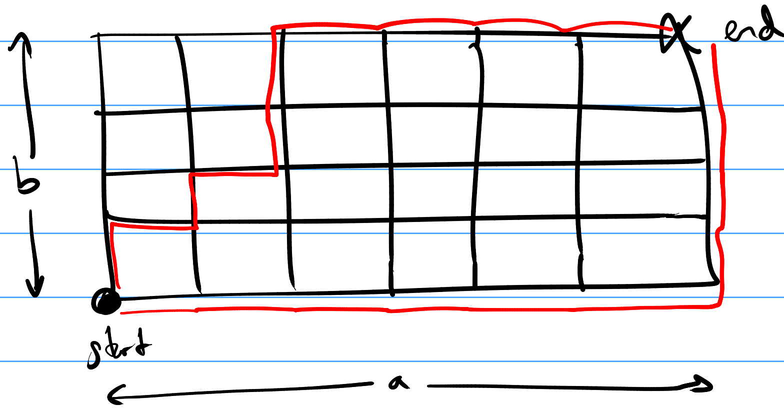
$X \subset \{1,2,\dots,n-1\}$

$f(X)$ Lay out n green balls. Put a red ball in the k'th slot for each $k \in X$.

So $\#$ compositions of n $= \#$ ball patterns $= \# \mathcal{P}(\{1,2,\dots,n-1\})$
$= 2^{n-1}$. $\square$

Lattice Paths

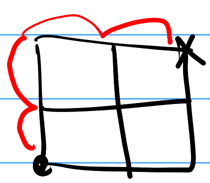
Consider rectangular lattice



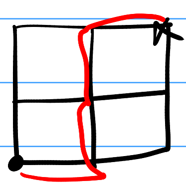
Defn admissible paths go from • to x along lattice lines only right and up.

Theorem There are $\binom{a+b}{a}$ admissible lattice paths through a rect. lattice of width a, height b.

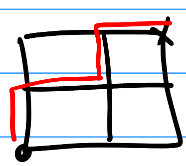
Proof bijection: admissible lattice paths $\longrightarrow$ strings containing a E's and b N's



NNEE $\{3,4\}$



ENNE $\{1,4\}$



NENE $\{2,4\}$

bijection: strings containing a E's & b N's $\longrightarrow P_a(\{1,2,\dots,a+b\})$

Write $y \in f(\mathrm{Str})$ iff there is E at position y in Str. $\square$