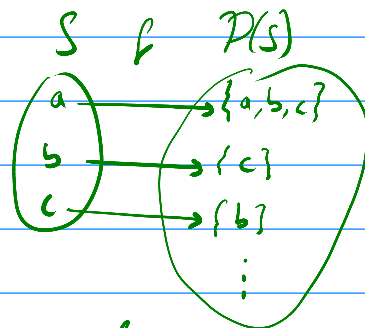
Cantor's Theorem 1 If S is a set, then $\nexists$ surj: $S \to \mathcal{P}(S)$.

Corollary 2 $\# S < \# \mathcal{P}(S)$.

Proof of Thm 1 Suppose $f : S \to \mathcal{P}(S)$. Then $\forall x \in S$, $f(x) \subset S$



$\forall x \in S$, we call x <u>f-happy</u> if $x \in f(x)$

<u>f-sad</u> if $x \notin f(x)$

So $a \in f(a)$, so a is f-happy

$b \notin f(b)$, $c \notin f(c)$, so b, c are f-sad.

Let $T = \{x \in S : f\text{-sad}\} \subset S$
$\in \operatorname{cod}(f)$

$\boxed{T \overset{?}{\in} \operatorname{im}(f).}$ Suppose $\exists\, y \in S$ s.t. $f(y) = T$. Then

$\boxed{y \in T}$ or $\boxed{y \notin T}$

$\Rightarrow y$ is f-sad by defn T	$\Rightarrow y$ is f-happy by defn T
$\Rightarrow y \notin f(y) = T$	$\Rightarrow y \in f(y) = T$
$\boxed{\Rightarrow y \notin T}$ ※	$\Rightarrow \boxed{y \in T}$ ※

$\Rightarrow$ no such y exists. So $T \notin \operatorname{im}(f)$. So $\operatorname{im}(f) \neq \operatorname{cod}(f)$

i.e. f not a surjection. $\square$

Proposition 3 Suppose $\# X = \# Y$. Then $\# \mathcal{P}(X) = \# \mathcal{P}(Y)$.

Moreover, if $\# X \leq \# Y$, then $\# \mathcal{P}(X) \leq \# \mathcal{P}(Y)$.

Proof Suppose $\exists$ inj $f : X \to Y$. Define $g : \mathcal{P}(X) \to \mathcal{P}(Y)$ by

g evaluated at S $g(S) = \{ f(x) : x \in S \} = f(S)$ image of S in f

Suppose $S, T \in \mathcal{P}(X)$ and $g(S) = g(T)$ (WTS $S = T$)

$\{f(x) : x \in S\} = f(S) = g(S) = g(T) = f(T) = \{ f(x) : x \in T\}$

$\forall x \in S \ \exists x' \in T$ s.t. $f(x) = f(x') \Rightarrow x = x'$
ie $\forall x \in S$, $x \in T$
$\Rightarrow S \subset T$

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$\Rightarrow S = T$. Hence g is inj.

Now suppose f is surj also. WTS g as defined above is surj.

Suppose $V \subset Y$ (ie $V \in \mathcal{P}(Y)$; $V \in \text{cod}(g)$).

If $y \in V$ then $\exists x \in X$ s.t. $f(x) = y$.

$\Rightarrow \exists U \subset X$ (ie $U \in \mathcal{P}(X)$; $U \in \text{dom}(g)$) s.t. $V = f(U) = g(U)$.
$\Rightarrow g$ is surj. $\square$