Theorem 1 $f: \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ \frac{1-n}{2} & n \text{ odd} \end{cases} \quad \text{is a bijection.}$$

Proof **Surj** If $z \in \mathbb{Z}$ then

$$\begin{cases} z > 0 \Rightarrow 2z \in \mathbb{N},\ 2z \text{ is even. } f(2z) = z \\ z \leq 0 \Rightarrow 1-2z \in \mathbb{N},\ 1-2z \text{ odd. } f(1-2z) = \frac{1-(1-2z)}{2} = z. \end{cases}$$

$\Rightarrow f$ is surj.

Inj Consider $x, y \in \mathbb{N}$ with $f(x) = f(y)$

f sends evens to positives, odds to nonpositives.

So x, y both even or x, y both odd

$\Rightarrow \frac{x}{2} = \frac{y}{2}$ | $\Rightarrow \frac{1-x}{2} = \frac{1-y}{2}$

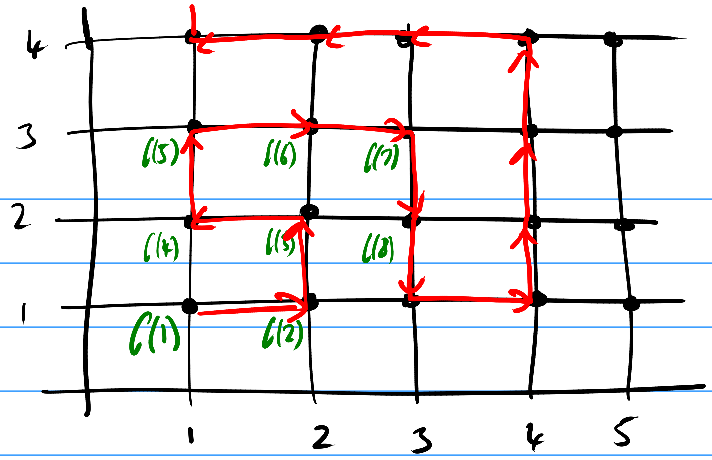
$\Rightarrow x = y$ | $\Rightarrow x = y$

$\Rightarrow f$ is an injection

So f is bijection. $\square$

Theorem 2 Define $f: \mathbb{N} \to \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ by

$f(1) = (1,1)$, $f(2) = (2,1)$, $f(3) = (2,2)$, $f(4) = (1,2)$, ...



Then f is a bijection.

Proof See the picture. $\square$