**Lemma 1** Let $T = \mathcal{P}(\mathbb{N}) \setminus \{x \in \mathcal{P}(\mathbb{N}) : x \text{ is finite}\}$. Then $\# T > \aleph_0$.

**Proof** Let $\boxed{S = \bigcup_{r \in \mathbb{N}} \mathcal{P}_r(\mathbb{N})}$ so that $T = \mathcal{P}(\mathbb{N}) \setminus S$.

We proved earlier $\forall r \in \mathbb{N}, \# \mathbb{N}^r = \aleph_0$. Let $\mathbb{N}^r_{\neq}$ be set of $r$-tuples of naturals with no repeated entries

$$(1,1), (1,3), (2,8), (4,4) \in \mathbb{N}^2$$

$$\cancel{(1,1)}, (1,3), (2,8), \cancel{(4,4)} \in \mathbb{N}^2_{\neq}$$

Define $g_r : \mathbb{N}^r_{\neq} \to \mathcal{P}_r(\mathbb{N})$ by

$$g_r((x_1, x_2, \dots, x_r)) = \{x_1, x_2, \dots, x_r\}$$

is a surjection because if $Y = \{y_1, y_2, \dots, y_r\}$ then

$$Y = g_r((y_1, y_2, \dots, y_r)).$$

So $$\# \mathcal{P}_r(\mathbb{N}) \underset{\substack{\uparrow \\ g_r \text{ surj}}}{\leq} \# \mathbb{N}^r_{\neq} \underset{\substack{\uparrow \\ \mathbb{N}^r_{\neq} \subset \mathbb{N}^r}}{\leq} \# \mathbb{N}^r = \aleph_0$$

So $S$ is a countable union of countable sets. So $\# S \leq \aleph_0$.

$\underset{\substack{\uparrow \\ \text{not countable}}}{\mathcal{P}(\mathbb{N})} = T \cup \underset{\substack{\uparrow \\ \text{countable}}}{S}$. If $T$ is countable then $T \cup S$ countable, which is false. So $\# T > \aleph_0$. $\square$

**Theorem 2** $\#\mathbb{R} > \aleph_0$.

**Proof** Let $T$ be as in Lemma 1. Define $g : T \to \mathbb{R}$ by

$$g(x) = \sum_{n \in x} \frac{1}{2^n}$$

We will show $g$ is an injection.

Suppose $x, y \in T$, $g(x) = g(y)$ and $x \neq y$. Because $x \neq y$ $\exists\, m \in \mathbb{N}$ s.t. $m$ belongs to one of $x, y$ but not the other.

Pick first such $m$. Suppose wlog $x \ni m \notin y$. Then

$$0 = g(x) - g(y) = \sum_{n \in x} \frac{1}{2^n} - \sum_{n \in y} \frac{1}{2^n}$$

$$= \left( \sum_{\substack{n \in x: \\ n < m}} \frac{1}{2^n} + \frac{1}{2^m} + \sum_{\substack{n \in x: \\ n > m}} \frac{1}{2^n} \right) - \left( \sum_{\substack{n \in y: \\ n < m}} \frac{1}{2^n} + \sum_{\substack{n \in y: \\ n > m}} \frac{1}{2^n} \right)$$

$$= \frac{1}{2^m} + \sum_{\substack{n \in x: \\ n > m}} \frac{1}{2^n} - \sum_{\substack{n \in y: \\ n > m}} \frac{1}{2^n}$$

$$\geq \frac{1}{2^m} + \sum_{\substack{n \in x: \\ n > m}} \frac{1}{2^n} - \boxed{\sum_{n = m+1}^{\infty} \frac{1}{2^n}} \quad = \frac{1}{2^m} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^m}(1)$$

geometric series lemma

$$= \sum_{\substack{n \in x: \\ n > m}} \frac{1}{2^n} > 0 \qquad ⨳$$

So no such $x, y$ exist; $g$ is inj. $\square$